1. Consider a system of particles moving on a two-dimensional substrate with randomly placed pinning centers. A pinning center $i$ located at the point $\mathbf{R}_i$ produces an attractive potential given by

$$v_i(\mathbf{r}) = -v_0 \text{ if } |\mathbf{r} - \mathbf{R}_i| \leq a; \quad v_i(\mathbf{r}) = 0 \text{ if } |\mathbf{r} - \mathbf{R}_i| > a.$$ 

The net pinning potential at the point $\mathbf{r}$ is given by

$$v(\mathbf{r}) = \sum_i v_i(\mathbf{r}).$$

The number of pinning centers per unit area of the surface is $n$, and there is no correlation between the locations of different pinning centers. Calculate $\langle v(\mathbf{r}) \rangle$ and $\langle v(\mathbf{r})v(\mathbf{r}') \rangle$ where $\langle \cdots \rangle$ represents an average over different realizations of the random pinning potential.

2. Consider a one-dimensional ferromagnetic Ising model defined by the Hamiltonian

$$H = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}(1 - \lambda n_i),$$

where $n_i = 0$ or $1$, and $\lambda < 1$ is a constant. The numbers $n_i$ may be regarded as the number of impurity atoms, i.e. if there is one impurity atom between sites $i$ and $i + 1$, then the interaction between the Ising spins $\sigma_i$ and $\sigma_{i+1}$ is weakened from $J$ to $J(1 - \lambda)$. The total number of impurities is $N' < N$.

(a) Assuming that the positions of the impurity atoms are not fixed, i.e. they can move to and fro, calculate the internal energy and the specific heat of the system as a function of the temperature $T$.

(b) Calculate the same quantities, assuming that the impurities are stationary and randomly distributed.

3. Consider a two-dimensional Ising model on a square lattice, defined by the Hamiltonian

$$H = - \sum_{<ij>} J_{ij} \sigma_i \sigma_j,$$

where the sum is over nearest-neighbor pairs and each $J_{ij}$ is an independent random variable that takes the values $+J$ and $-J$ with probabilities $p$ and $(1 - p)$, respectively.

(a) Calculate the probability of an elementary plaquette of the lattice to be frustrated.

(b) Calculate the probability of two adjacent plaquettes to be frustrated simultaneously. Is this probability equal to the square of the probability calculated in part (a)?
(c) The centers of the plaquettes of the original square lattice form another square lattice which is called the dual of the original lattice. Assume that a site on the dual lattice is “occupied” if the corresponding plaquette is frustrated. Estimate the value of $p$ at which these “occupied” sites would percolate on the dual lattice.

(d) Consider the model in which each $J_{ij}$ is given by $J_{ij} \tau_i \tau_j$ where each $\tau_i$ is an independent quenched random variable taking the values $+1$ and $-1$ with equal probability. Does this model have any frustration? Find the ground state(s) of this model, which is known as the Mattis model.

4. Consider 4 Ising spins, $S_1, S_2, S_3,$ and $S_4,$ forming a cluster as shown below.

![Diagram of 4 Ising spins forming a cluster](image)

These spins are coupled by nearest- and next-nearest-neighbor interactions $\{J_{ij}\},$ each of which is an independent quenched random variable, taking the values $+J$ and $-J$ with probability $p$ and $(1 - p),$ respectively. Using exact enumeration on a computer, calculate the disorder average of $E_g,$ the ground state energy of this system, as a function of $p.$ Plot the result as a function of $p$ in the range $0 \leq p \leq 1.$

5. Consider the Edwards-Anderson model of spin glass for Ising spins on a simple cubic lattice, defined by the Hamiltonian

$$H = - \sum_{<ij>} J_{ij} \sigma_i \sigma_j,$$

where the sum is over nearest-neighbor pairs of lattice sites and each $J_{ij}$ is an independent random variable distributed according to

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi}J} \exp(-J_{ij}^2/2J^2).$$

(a) Use the replica method to obtain the effective Hamiltonian of the system in terms of the replicated Ising variables $\{\sigma_i^\alpha\}, i = 1, \ldots, N, \alpha = 1, \ldots, n,$ where $N$ is the number of lattice sites and $n$ is the number of replicas.
(b) The magnetization $M$ of the system as a function of the magnetic field $h$ can be written as

$$M = \chi_1 h + \frac{1}{6} \chi_3 h^3 + \text{higher powers of } h.$$ 

Derive expressions for the usual linear susceptibility $\chi_1$ and the nonlinear susceptibility $\chi_3$ in terms of correlation functions of (i) the original spin variables $\{\sigma_i\}$; and (ii) the replicated spin variables $\{\sigma_i^a\}$. 
