1. Real-space renormalization group calculation for site percolation: Consider the site percolation problem on a two-dimensional triangular lattice. Divide the sites of the original lattice $L$ into groups of three as shown below.

The three sites in a group form an elementary triangle of the original lattice $L$. The centres of these triangles form another triangular lattice $L'$ whose lattice constant is $b = \sqrt{3}$ times larger than the lattice constant of $L$. The site percolation problem with occupation probability $p$ on the original lattice $L$ can be mapped to one on $L'$ by assuming that a site in $L'$ is occupied if a majority of the sites of the corresponding triangle (i.e. at least two of the three sites) are occupied.

(a) Calculate the occupation probability $p'$ of a site on $L'$ as a function of $p$. The function $p'(p)$ defines a renormalization group transformation $R$ in which the length scale changes by a factor $b = \sqrt{3}$.

(b) Show that the renormalized occupation probability flows to $0(1)$ under repeated applications of $R$ if $p < p^*(p > p^*)$. Determine the value of $p^*$ and give a physical interpretation of this "special" value of $p$.

(c) Define the correlation length for $p < p^*(p > p^*)$ as the length scale at which the renormalized occupation probability is $0.01(0.99)$. Show that the correlation length defined in this way diverges as $p$ approaches $p^*$. Calculate the value of the exponent $\nu$ that describes this divergence.

2. Consider the bond percolation problem on a $n \times n$ square lattice with open boundary conditions. Write a computer program to generate typical configurations of occupied bonds for different values $p$, the probability of a bond being occupied. Also, write a computer program to determine whether both pairs of opposite sides (top-bottom and left-right) are connected by occupied bonds in a particular configuration. Use these programs to determine the percolation probability (the probability that both pairs of opposite sides are connected) as a function of $p$, and plot the results for $n = 3, 4$. 

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3. Site percolation on a Bethe lattice: Consider an infinite Bethe lattice with coordination number \( z \) in which each site is occupied with probability \( p \) and vacant with probability \( 1 - p \). Obtain an explicit expression for the probability that an occupied site is a member of an infinite cluster, and calculate the critical value \( p_c \) of \( p \) below which this probability is zero.

4. Consider the bond-diluted Ising ferromagnet on a square lattice. Assume that each nearest-neighbor “bond” is present with probability \( p \) and absent with probability \( 1 - p \). The interaction strength of a bond that is present is \( J \). Consider a site and its four neighbors. Assume that each neighbor has a magnetization \( m \). Calculate the average (over the occupation probabilities of the four bonds that connect the central site with its nearest neighbors) value of the magnetization of the central site. Obtain a self-consistent equation for \( m \) by equating this average value with \( m \). Solve this self-consistent equation to determine the transition temperature \( T_c \) as a function of \( p \).

5. Consider a \( d \)-dimensional disordered system in which the disorder is perfectly correlated in \( d' \) dimensions \( (d' < d) \). An example of such a system (with \( d = 2 \) and \( d' = 1 \)) would be a spin model on a square lattice in which the exchange interaction varies randomly in one direction and the (random) values of the exchange interaction are exactly repeated in the other direction. Derive a generalized Harris criterion for the occurrence of a continuous phase transition in such a system.