Slow quench dynamics in classical systems

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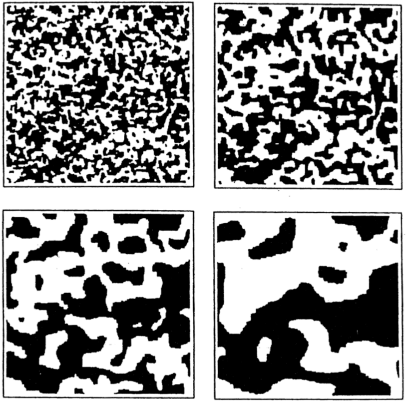
Instantaneous quench: phase ordering (Bray 2002)

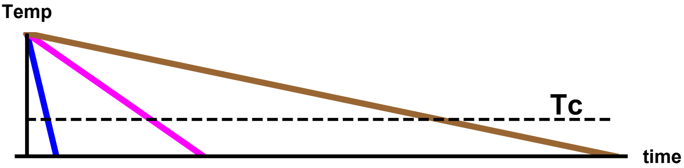
- Prepare the system in (disordered) state above T_c
- Lower the temp to/below T_c instantly
- How does the system approach the stationary state?

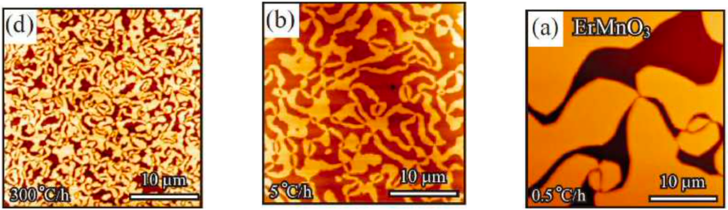
$$\ell \sim t^{1/z_{neq}}$$



Slow quench: residual defects

- Prepare the system in (disordered) state above T_c
- Cool to $T \leq T_c$ in finite time, τ; no further evolution





Rapid quench Slow quench Equil state

(Chae et al. 2012)

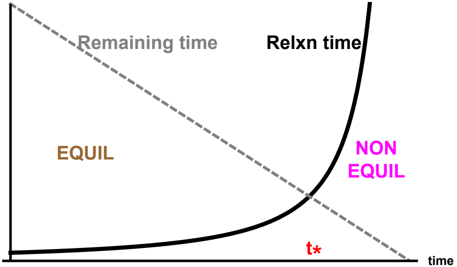
Kibble-Zurek argument (Kibble 1976, Zurek 1985)

- How does excess defect density decay with quench time?

$$D(\tau) - D_{eq}(T(\tau)) \sim \tau^{-?}$$

- Away from CP: relxn time $\sim \mathcal{O}(1)$, remain time $\sim \mathcal{O}(\tau)$

  Close to CP: relxn time $\sim \xi^{z_{ss}}(t)$, remain time<relxn time



- Time scale t_* separates these two behavior

$$\tau - t_* \sim \xi^{z_{ss}}(t_*) \sim |T_c - T(t_*)|^{-\nu z_{ss}}$$

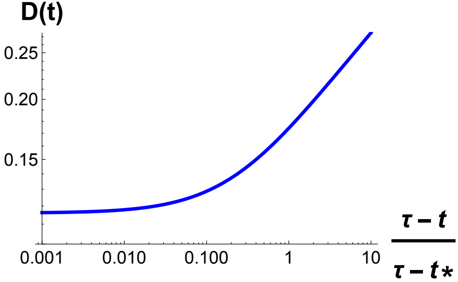
For e.g., cooling protocol ending at T_c algebraically

$$|T_c - T(t)| \sim \left(1 - \frac{t}{\tau}\right)^{\beta}$$

  - KZ time scale, $\tau - t_* \sim \tau^{\frac{\beta \nu z_{ss}}{1+\beta \nu z_{ss}}}$
  - KZ length scale, $\xi(t_*) \sim \tau^{\frac{\beta \nu}{1+\beta \nu z_{ss}}}$
  - Interpolates between infinitesimally slow and fast quench

Residual density of defects (Zurek 1985)

- Assume the dynamics are 'frozen' for $t > t_*$



Pass on the equilibrium defects at t_* to τ

$$\delta D(\tau) \sim \delta D_{eq}(t_*) \sim \xi^{-d}(t_*) \sim \tau^{\frac{-d\beta\nu}{1+\beta\nu z_{ss}}}$$

- Scalings verified numerically and experimentally

(review, Campo & Zurek 2014)

‘Frozen dynamics’? (Biroli, Cugliandolo, Sicilia 2010)

- Model $t > t_*$ dynamics as instantaneous quench problem.

  Coarsening occurs for $t_* < t < \tau$

$$\ell(t) \sim (t - t_*)^{1/z_{neq}}$$

- Defect density at the end of quench scales differently

$$\delta D(\tau) \sim \ell^{-d}(\tau) \sim (\tau - t_*)^{-d/z_{neq}} \sim \tau^{-\frac{d\beta\nu(z_{ss}/z_{neq})}{1+\beta\nu z_{ss}}}$$

- Scaling numerically verified in systems where $z_{ss} \neq z_{neq}$

Some questions

- Numerical verification of scaling argument(s)

  Derive KZ scales from dynamical eqns? Analytical expressions for the defect density? Scaling function?

- Focus on effect of coarsening at the end of quench

  What is the role of coarsening near the CP?

  Ans: Inst quench model good only in a parameter regime

- Work with equilibrium models

  (Classical) nonequilibrium models?

Plan

- Models
    - 1D Ising model with Glauber dynamics
    - Mean field zero range process
    - 1D asymmetric exclusion process
- Final quench to the critical point

Glauber Ising chain (Glauber 1963)

- Equilibrium state of ferromagnetic Ising model

$$H = -\sum_{i=1}^{L} \sigma_i \sigma_{i+1} \ , \ \sigma_i \pm 1$$

No phase transition

- Dynamical evolution via single-spin flips

$$\dot{\Psi}(\{\sigma\}) = \sum_j W(-\sigma_j \to \sigma_j)\Psi(\{-\sigma_j\}) - W(\sigma_j \to -\sigma_j)\Psi(\{\sigma\})$$

where transition rates satisfy detailed balance

$$W(\sigma_j \to -\sigma_j) = 1 - \tanh\left(\frac{2}{T}\right) \frac{\sigma_j(\sigma_{j-1} + \sigma_{j+1})}{2}$$

Relevant dynamical exponents

- Equil dynamics at low temp due to unbiased random walk of domain walls (no creation/annihilation allowed)

$$\ldots \uparrow\uparrow\uparrow\uparrow \downarrow \uparrow\uparrow\uparrow\uparrow\uparrow \ldots$$

$$\xi \sim \sqrt{t}$$

- Nonequil dynamics following inst quench to low temp due to domain of length ℓ merging over ℓ^2 time

$$\ell(t) \sim \sqrt{t}$$

- Spoiler alert! As $z_{ss} = z_{neq} = 2$, expect KZ scaling hold for defect density

Spin-spin correlation function

- $G_k(t) = \langle \sigma_i(t)\sigma_{i+k}(t)\rangle$ obeys closed linear equation

$$\dot{G}_k = -2G_k + \gamma(T)(G_{k-1} + G_{k+1}) \ , \ \gamma = \tanh\left(\frac{2}{T}\right)$$

- In equilibrium state: $G_k \overset{T\to 0}{\sim} e^{-k/\xi}$

  Equil corr length, $\xi \overset{T\to 0}{\sim} e^{2/T}$ (not algebraically)

- Here, interested in domain walls $\uparrow\downarrow$

$$D(t) = \frac{1 - G_1(t)}{2}$$

Slow cooling dynamics

- For time-depn temperature

$$\dot{G}_k = -2G_k + \gamma(t)(G_{k-1} + G_{k+1}) \;, \;\; k = 1, \ldots, L$$

with $G_1(t) = G_L(t) = 1$, $G_k(0) = 0$.

- Exact soln known for any $\gamma(t)$! (Reiss'80, Brey+Prados'94)

For an infinitely large system,

$$\mathcal{G}_k(t) - G_k(t) = \int_0^\pi dq \phi_k(q) \int_0^t dt' e^{-2\int_{t'}^t dy(1-\gamma(y)\cos q)} \sum_{m=1}^{\infty} \phi_m(q) \dot{\mathcal{G}}_m(t')$$
$$+ \int_0^\pi dq \phi_k(q) e^{-2\int_0^t dy(1-\gamma(y)\cos q)} \sum_{m=1}^{\infty} \phi_m(q)(\mathcal{G}_m(0) - G_m(0)) \;,$$

where $\phi_k(q) \sim \sin(kq)$

$$\dot{G}_k = -2G_k + \gamma(t)(G_{k-1} + G_{k+1}) \ , \ \ k = 1, ...$$

Expand as linear combination of orthonormal eigenvectors,

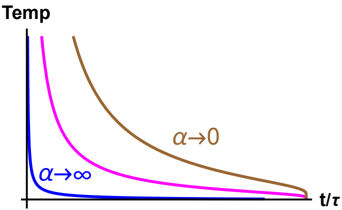
$\phi_k(q)$ that obey: $-\lambda(\gamma)\phi_k = -2\phi_k + \gamma(\phi_{k-1} + \phi_{k+1})$

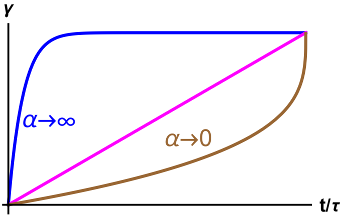
$$\begin{aligned} G_{k,eq}(t) - G_k(t) &= \int dq \phi_k(q) a(q,t) \\ \dot{G}_{k,eq}(t) &= \int dq \phi_k(q) b(q,t) \end{aligned}$$

(Brey & Prados 1994)

Cooling protocols

- Cool from $T \to \infty$ to $T = 0$ in time $\tau \gg 1$





$$T(t) \overset{t\to\tau}{\sim} \frac{-\ln(1-\frac{t}{\tau})}{\alpha} \qquad \gamma(t) = 1 - [1 - \left(\frac{t}{\tau}\right)^{\alpha}]$$

- Algebraic cooling also studied but not 'interesting'

Low temp dynamics

- Scaling ansatz: $G_{k,eq}(t) - G_k(t) = F(K, Z)$

$$K = k\tau^{-a} \ , \ Z = (\tau - t)\tau^{b}$$

  for large k, τ and $t \to \tau$

- Plug in the eqn for corr fn; simple power counting yields

$$K = \frac{k}{\xi(t_*)} = \frac{k}{\tau^{\frac{\alpha}{2(1+\alpha)}}} \ , \ Z = \frac{\tau - t}{\tau - t_*} = \frac{\tau - t}{\tau^{\frac{\alpha}{(1+\alpha)}}}$$

- Agrees with KZ argument for t_*; here, derived from dyn eqns and obtained full corr fn, $F(K, Z)$

Residual defect density

- At the end of quench,

$$D(\tau) = \frac{1}{\tau^{\frac{\alpha}{2(1+\alpha)}}} \frac{1}{2\sqrt{\pi}} \left(\frac{2}{1+\alpha} \right)^{\frac{1}{2(1+\alpha)}} \Gamma\left(\frac{1+2\alpha}{2+2\alpha} \right)$$

- Scaling same as KZ and Biroli et al. (since $z_{ss} = z_{neq}$)
- Prefactors show $D(\tau) < D_{eq}(t_*)$
  - Supports coarsening? (but t_* is a scale!)
  - Analytical (Krapivsky 2010), Simulation (Jeong et al. 2020)

Testing the coarsening proposal

For finite time quench

$$G_k(\tau) - G_k(t) \overset{t\to\tau}{=} \begin{cases} \dfrac{4K\alpha}{\sqrt{\pi}(1-2\alpha)} Z^{\alpha+\frac{1}{2}} & , \quad \alpha < 1/2 \\ \dfrac{2K\alpha}{\sqrt{\pi}} \ln(1/Z) Z & , \quad \alpha = 1/2 \\ \dfrac{K}{2\sqrt{\pi}} \left(\dfrac{2}{1+\alpha}\right)^{\frac{3}{2(1+\alpha)}} \Gamma\left(\dfrac{2\alpha-1}{2\alpha+2}\right) Z & , \quad \alpha > 1/2 \end{cases}$$

For instantaneous quench (Glauber 1963)

$$G_k(T=0) - G_k(t) = \frac{KZ}{2\sqrt{\pi}} \ , \quad \alpha \to \infty$$

Inst quench good in nonequil regime if quench is not too slow

Algebraic cooling of Ising model

- Here,

$$T(t) \overset{t\to\tau}{\sim} 4\,(1-x)^{\beta}$$

$$\gamma(t,\tau) = 1 - \exp\left[1-(1-x)^{-\beta}\right]$$

- Scaling variables match with KZ argument

$$K = k\sqrt{\frac{(\ln\tau)^{1/\beta}}{\tau}}\ ,\ \ Z = (1-x)(\ln\tau)^{1/\beta}\ .$$

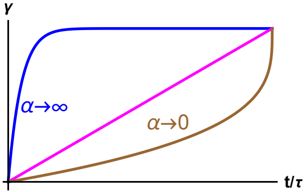
- As this cooling is faster than log cooling, corresponds to $\alpha\to\infty$ <u>for any β</u>:

$$G_k(T=0) - G_k(t) = \frac{KZ}{2\sqrt{\pi}}$$

Summary

- More generally, close to CP, we expect

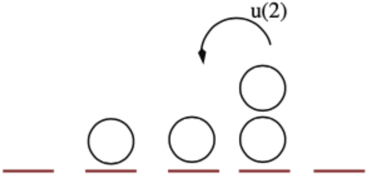
$$F(K,Z) = \begin{cases} A_1(K,\alpha) + A_2(K,\alpha) Z^{a<1} \ , & \alpha < \alpha_c \\ \hat{A}_1(K,\alpha) + \hat{A}_2(K,\alpha) Z \ , & \alpha > \alpha_c \end{cases}$$



- A_i's depn on cooling scheme (nonuniversal); comparison with that at t_* not satisfactory. Instead look at <u>scaling with Z</u> to test 'coarsening proposal'
- Inst quench good if finite time quench is fast enough

Zero range process (review, Evans & Hanney 2005)

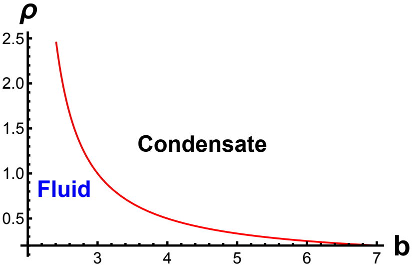
- Each site can be occupied by any number of particles.
- One particle hops to a nearest neighbor at rate $u(m)$

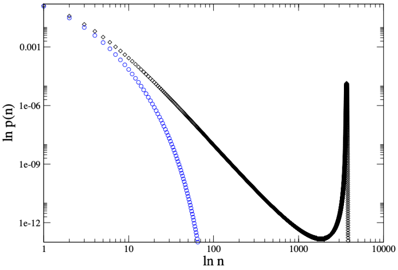


- Hops can be symm/asymm; any dimension
- Exact stationary distribution known for general model

Condensation transition

$$u(m) = 1 + \frac{b}{m}\ ,\ \ b \geq 0$$





(Evans & Hanney 2005)

Phase transition for $b > 2$

Slow quench

- Anneal the system from $b = 0$ to $b = b_c$ in time τ

$$u(m, t) = 1 + \frac{b(t)}{m} \; , \; b \geq 0$$

$$b(t) = b_c \left[1 - \left(1 - \frac{t}{\tau} \right)^{\alpha} \right]$$

- Mean field geometry: Scaling ansatz in master equation

$$\frac{P(m,t)}{P_{ss}(m,t)} = Q(K = m\tau^{-a}, Z = (\tau - t)\tau^{b})$$

Consistent with KZ argument (since $z_{ss} = z_{neq} = 2$)

But scaling function's eqn doesn't seem solvable

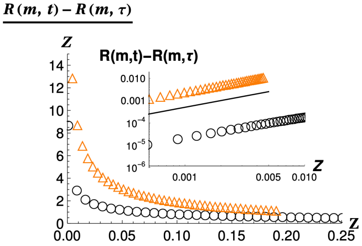
Mean field zero range process

- Instantaneous quench (Godréche 2003)

$$R(m,t) \equiv \frac{P(m,t)}{P_{ss}(m,t)} = c_1(M) + c_2(M) Z$$

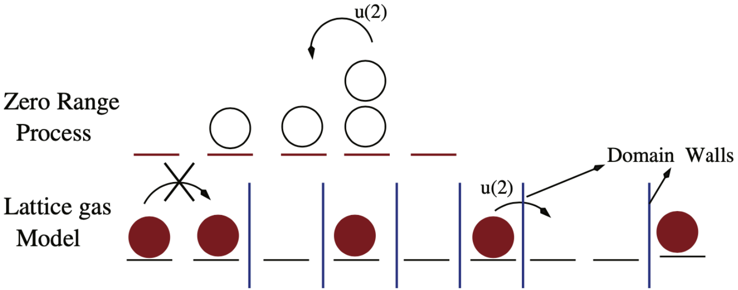
- Slow quench simulations (for fixed m)

$$R(m,t) - R(m,\tau) \sim \begin{cases} Z^a, \alpha < 1 \\ Z, \alpha > 1 \end{cases}$$



Slow quench in nonequil models

- 1D asymmetric exclusion process



- Numerically found KZ scaling for defect density does not hold; consistent with Biroli et al. ($z_{ss} = 3/2, z_{neq} = 3$)
- Scaling function's behavior?

Open questions

- For class of models with $z_{ss} = z_{neq}$ where KZ and Biroli et al. coincide, (Priyanka, Chatterjee, Jain 2021)

$$F(K,Z) = \begin{cases} A_1(K,\alpha) + A_2(K,\alpha) Z^a \ , & \alpha < \alpha_c \\ \hat{A}_1(K,\alpha) + \hat{A}_2(K,\alpha) Z \ , & \alpha > \alpha_c \end{cases}$$

  - True for models with $z_{ss} \neq z_{neq}$? Perhaps study Kawasaki Ising chain/1D Excl process (eqns don't close)
  - Simple, general argument for α_c?
  - Quenches to ordered phase? Other annealing schemes?

- Analytical results for high-dim equil models? Models with nonequilibrium SS (Priyanka, Jain 2016)?